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A STUDY ON THE STABILITY OF 3-FIELD FINITE ELEMENTS BY THE THEORY OF ZERO ENERGY MODES

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Abstract—The theory of zero energy modes (ZEM) for 3-field (ε–σ–$\mathbf{u}$) finite elements is presented. A systematic study has been made on the numerical stability of the mixed element and the hybrid element respectively.

1. INTRODUCTION

The investigation into the stability of the numerical solutions has been a major problem in the development of multi-field (or multi-variable) elements. It is true that some mathematical basis for this problem has been established (Babuska and Aziz, 1972; Brezzi, 1974; Girault and Raviart, 1986). In particular, an inf sup condition with the constant independent of h was presented, which is necessary and sufficient for convergence in the precise sense and guarantees that the resulting discrete system has a unique solution (Arnold, Babuska and Osborn, 1984). However, because of its abstract concept and the complex analysis, it has not been widely accepted by engineers, who prefer to use some simpler stability conditions as guidelines in the development and application of multi-field elements (Tong and Pian, 1969; Zienkiewicz *et al.*, 1986; Zienkiewicz and Lefebure, 1987).

From the mechanics point of view, the stability problem is really connected with ZEM. In this paper we call the element without ZEM a stable one, and a finite element system is regarded as stable if it does not contain any ZEM and that the resulting discrete equations possess a unique solution. In order to guarantee the stability of hybrid elements, Pian and Chen (1983) proposed a method for the suppression of zero energy displacement modes (ZEM(u)). The method was generalized to 2-field mixed/hybrid elements by Wu (1990), who also discussed in greater detail the definition, the analysis and control principles for ZEM. In this paper, the theory of ZEM will be further generalized to 3-field elements. Unlike the 2-field elements, two types of ZEM (ZEM(σ) for stress and ZEM(u) for displacements) can appear simultaneously for a 3-field element. Furthermore, the stability requirement for the mixed element is somewhat lower than that for the hybrid element in the 3-field situation, and accordingly the stability problem for the mixed elements will be considered first in our discussions.

2. STABILITY CONDITION OF 3-FIELD MIXED ELEMENTS

We denote the strain, stress and displacement trial functions of a 3-field mixed finite element as ε, σ and $\mathbf{u}$ respectively. Correspondingly, there exist three sets of element trial functions: $\mathrm{E} \equiv \{\varepsilon\}$, $\Sigma \equiv \{\sigma\}$ and $U \equiv \{\mathbf{u}\}$. In order to make the discussion easier on the ZEM, it would be appropriate to eliminate the rigid-body displacements of an element from $\mathbf{u}$, and then the remaining ones in $\mathbf{u}$, which correspond to the non-zero element strain, are denoted by $\mathbf{u}_* \in U_* \equiv \{\mathbf{u}_*\}$. The Hu–Washizu energy functional and its various modified or generalized forms (Bufler, 1979; Oden and Raddy, 1983) may all be expressed, for an individual element without rigid-body displacement, as

$$\pi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}_*) = \tfrac{1}{2}\langle c\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle + I_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) + I_2(\boldsymbol{\sigma}, \mathbf{u}_*). \tag{1}$$

Here the first term, i.e. the element strain energy, $\frac{1}{2}\langle c\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle = \frac{1}{2}\int_{v^e} \boldsymbol{\varepsilon}^T c\boldsymbol{\varepsilon}\, dv$ (v^e = element domain, c = elastic matrix) is a positive definite quadratic form of $\boldsymbol{\varepsilon}$, while the second term $I_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = -\int_{v^e} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dv$ is a bilinear integral term for the pair of variables $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$, and the third term $I_2(\boldsymbol{\sigma}, \mathbf{u}_*)$ is an algebraic sum of some bilinear integral terms for the pair $(\boldsymbol{\sigma}, \mathbf{u}_*)$ (e.g. $\int_{v^e} \boldsymbol{\sigma}^T(\mathbf{D}\mathbf{u}_*)\, dv$ etc., where $\mathbf{D}$ = strain differential operator). Other terms which are related to the applied load and prescribed displacement, which are independent of our ZEM analysis, have not been included in the functional (1).

The definition of ZEM for 2-field finite elements has been presented by Wu (1990), and the mechanical essence is to examine whether the element trial function is able to provide the element with an energy contribution. Now in the more general case of the 3-field mixed element we have the following:

Definition. A non-zero strain $\ddot{\varepsilon} \in \mathrm{E}$ is said to be the zero energy strain mode ZEM(ε) if the functional increment

$$\Delta\pi[\varepsilon] \equiv \pi(\varepsilon + \ddot{\varepsilon}, \sigma, u_*) - \pi(\varepsilon, \sigma, \mathbf{u}_*) = 0 \quad \forall(\sigma, \mathbf{u}_*) \in \Sigma \times U_* \tag{2}$$

and a non-zero stress $\ddot{\boldsymbol{\sigma}} \in \Sigma$ is said to be the zero energy stress mode ZEM(σ) if the functional increment

$$\Delta\pi[\ddot{\sigma}] \equiv \pi(\varepsilon, \sigma + \ddot{\sigma}, \mathbf{u}_*) - \pi(\varepsilon, \sigma, \mathbf{u}_*) = 0 \quad \forall(\varepsilon, \mathbf{u}_*) \in \mathrm{E} \times U_* \tag{3}$$

and a non-zero displacement $\ddot{\mathbf{u}} \in U_*$ is said to be the zero energy displacement mode ZEM(u) if the functional increment

$$\Delta\pi[\ddot{\mathbf{u}}] \equiv \pi(\varepsilon, \sigma, \mathbf{u}_* + \ddot{\mathbf{u}}) - \pi(\varepsilon, \sigma, \mathbf{u}_*) = 0 \quad \forall(\varepsilon, \sigma) \in \mathrm{E} \times \Sigma. \tag{4}$$

In accordance with these definitions, it is evident that if

$$\forall(\boldsymbol{\sigma}, \mathbf{u}_*) \in \Sigma \times U_*, \quad \Delta\pi[\varepsilon] = 0 \quad \Rightarrow \quad \varepsilon = 0 \tag{5}$$

the mixed element has no ZEM(ε); and

$$\forall(\varepsilon, \mathbf{u}_*) \in \mathrm{E} \times U_*, \quad \Delta\pi[\sigma] = 0 \quad \Rightarrow \quad \sigma = 0 \tag{6}$$

the mixed element has no ZEM(σ). Finally, if

$$\forall(\varepsilon, \boldsymbol{\sigma}) \in \mathrm{E} \times \Sigma, \quad \Delta\pi[\mathbf{u}_*] = 0 \quad \Rightarrow \quad \mathbf{u}_* = 0 \tag{7}$$

the mixed element has no ZEM(u). Obviously the 3-field mixed element without any ZEM becomes a stable one when the conditions (5)–(7) are satisfied simultaneously.

Considering the functional (1), for an arbitrary strain increment ε' to be independent of ε,

$$\Delta\pi[\varepsilon'] = \tfrac{1}{2}\langle c(\varepsilon + \varepsilon'), (\boldsymbol{\varepsilon} + \varepsilon')\rangle - \tfrac{1}{2}\langle c\varepsilon, \varepsilon\rangle + I_1(\varepsilon', \sigma). \tag{8}$$

Note that since the first two terms in (8) are positive definite quadratic forms of the strain, it follows that

$$\forall(\boldsymbol{\sigma}, \mathbf{u}_*) \in \Sigma \times U_*, \quad \Delta\pi[\varepsilon'] = 0 \quad \Rightarrow \quad \varepsilon' = 0$$

and therefore condition (5) is constantly satisfied by the mixed element. By the way, the same analysis is also valid for the 3-field hybrid element. Thus we can assert that *the 3-field finite element based on the Hu–Washizu principle has no ZEM(ε)*. On the other hand, it is easy to find that, for the mixed element based on (1),

$$\Delta\pi[\sigma] = I_1(\varepsilon,\sigma)+I_2(\sigma,\mathbf{u}_*)$$

and

$$\Delta\pi[\mathbf{u}_*] = I_2(\sigma,\mathbf{u}_*).$$

Thus the conditions (6) and (7) can be stated in another manner:

$$\forall(\varepsilon,\mathbf{u}_*)\in \mathrm{E}\times U_*,\quad I_1(\varepsilon,\sigma)+I_2(\sigma,\mathbf{u}_*) = 0 \quad\Rightarrow\quad \sigma = 0 \tag{9}$$

and

$$\forall\sigma\in\Sigma,\quad I_2(\sigma,\mathbf{u}_*) = 0 \quad\Rightarrow\quad \mathbf{u}_* = 0 \tag{10}$$

respectively. In conclusion, the satisfaction of the above conditions will prevent the appearance of ZEM(σ) and ZEM(u); (9) and (10) are therefore *the stability conditions of the 3-field mixed element*.

3. MIXED FORMULATION AND KEEPING RANK CONDITION

The trial functions of a 3-field mixed element are defined as follows. The element strain

$$\varepsilon = \psi\alpha,\quad \alpha = \text{element strain parameter}. \tag{11}$$

Here α may be the nodal strain value or the internal parameter of the element strain. We will see later that α can always be eliminated at the element level without any extra supplementary condition. So we prefer to take α as the internal parameter of the mixed element in order to reduce the size of resulting system equations. Similarly, the element stress is expressed as

$$\sigma = \phi\beta,\quad \beta = \text{element nodal stress} \tag{12}$$

and the element displacement as

$$\mathbf{u} = \mathbf{N}\mathbf{q},\quad \mathbf{q} = \text{element nodal displacement}. \tag{13}$$

We assume that at least r nodal displacements must be constrained in $\mathbf{q}$ to prevent the element rigid-body motion, and the remaining ones in $\mathbf{q}$ are denoted by $\mathbf{q}_*$. Accordingly, instead of (13) the displacement trial function will be rewritten as

$$\mathbf{u}_* = \mathbf{N}_*\mathbf{q}_*. \tag{14}$$

By means of (11), (12) and (14), the functional (1) may be discretized and expressed as

$$\pi(\alpha,\beta,\mathbf{q}_*) = \tfrac{1}{2}\alpha^{\mathrm{T}}\mathbf{A}\alpha+\alpha^{\mathrm{T}}\mathbf{F}\beta+\beta^{\mathrm{T}}\mathbf{G}_*\mathbf{q}_*. \tag{15}$$

From the stationary condition $\delta\pi = 0$, we obtain a set of discrete equations

$$\left.\begin{aligned} \mathbf{A}\alpha+\mathbf{F}\beta &= 0\\ \mathbf{F}^{\mathrm{T}}\alpha+\mathbf{G}_*\mathbf{q}_* &= 0\\ \mathbf{G}_*^{\mathrm{T}}\beta &= 0 \end{aligned}\right\}. \tag{16}$$

In accordance with the positive definite quadratic form $\langle c\varepsilon,\varepsilon\rangle$ in (1), the symmetric matrix $\mathbf{A} = \int_{v^e}\psi^{\mathrm{T}}c\psi\,\mathrm{d}v$ must be a positive definite one (provided that ψ does not violate the well-known independence requirement on the basis function). As a result the strain parameter α can be eliminated from (16), and we obtain the following mixed discrete equations

$$\begin{bmatrix} -\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F} & \mathbf{G}_* \\ \mathbf{G}_*^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta} \\ \mathbf{q}_* \end{Bmatrix} = 0. \tag{17}$$

Here $\begin{Bmatrix} \boldsymbol{\beta} \\ \mathbf{q}_* \end{Bmatrix}$ plays the role of *the basic discrete parameters* of a 3-field mixed element without the rigid body DOF, while $\boldsymbol{\alpha}$ becomes a *related parameter*. Obviously the resulting discrete matrix in (17) is non-positive definite, but it is invertible provided that the mixed element has no ZEM.

With reference to formulation (15), the stability condition (9) is now

$$\forall(\boldsymbol{\alpha}, \mathbf{q}_*) \in \{\boldsymbol{\alpha}\} \times \{\mathbf{q}_*\}, \quad \boldsymbol{\alpha}^T\mathbf{F}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{G}_*\mathbf{q}_* \equiv \begin{bmatrix} \boldsymbol{\alpha} \\ \mathbf{q}_* \end{bmatrix}^T \begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix} \boldsymbol{\beta} = 0 \quad \Rightarrow \quad \boldsymbol{\beta} = 0 \tag{18}$$

or equivalently,

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix} \boldsymbol{\beta} = 0 \quad \Rightarrow \quad \boldsymbol{\beta} = 0. \tag{19}$$

At the same time condition (10) can be expressed as

$$\forall \boldsymbol{\beta} \in \{\boldsymbol{\beta}\}, \quad \boldsymbol{\beta}^T\mathbf{G}_*\mathbf{q}_* = 0 \quad \Rightarrow \quad \mathbf{q}_* = 0$$

or equivalently,

$$\mathbf{G}_*\mathbf{q}_* = 0 \quad \Rightarrow \quad \mathbf{q}_* = 0. \tag{20}$$

(19) and (20), which ensure that the element has no ZEM(σ) and ZEM(u) respectively, are to be called *the keeping rank conditions* of the 3-field mixed element, which are the necessary and sufficient conditions for guaranteeing the absence of ZEM at the element level.

If we designate

$$n_\alpha = \dim(\boldsymbol{\alpha})$$
$$n_\beta = \dim(\boldsymbol{\beta})$$
$$n_{q_*} = \dim(\mathbf{q}_*)$$

then there exist two necessary conditions for meeting (19) and (20) respectively. They are

$$n_\alpha + n_{q_*} \geq n_\beta \tag{21}$$

and

$$n_\beta \geq n_{q_*}. \tag{22}$$

In short, the positive integers n_α, n_β and n_q have to satisfy *the parameter matching conditions*

$$n_\alpha + n_{q_*} \geq n_\beta \geq n_{q_*}. \tag{23}$$

They are consistent with the result given by Zienkiewicz and Lefebure (1987). The matching conditions are very useful in the design of the mixed models even though they are just a set of necessary conditions for the element stability.

4. ZEM(σ) EXHIBITION AND CONTROL

When condition (19) cannot be satisfied, n_0 ZEM(σ) will appear, and

$$n_0 = n_\beta - \text{rank}\left(\begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix}\right). \tag{24}$$

We denote the general solution of the homogeneous equation in (19) by

$$\boldsymbol{\beta} = \mathbf{T}_\beta \mathring{\boldsymbol{\beta}} \tag{25}$$

where $\mathring{\boldsymbol{\beta}}$ is composed on n_0 arbitrary parameters and can be expressed as

$$\mathring{\boldsymbol{\beta}} = \begin{Bmatrix} \beta_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \beta_2 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + \cdots + \begin{Bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \beta_{n_0} \end{Bmatrix} = \sum_i^{n_0} \mathring{\boldsymbol{\beta}}(i). \tag{26}$$

In terms of (25), (26) and the stress trial function (12), the n_0 ZEM(σ) can be independently exhibited by

$$\mathring{\boldsymbol{\sigma}}(i) = \boldsymbol{\phi}\mathbf{T}_\beta \mathring{\boldsymbol{\beta}}(i), \quad i = 1, 2, \ldots, n_0. \tag{27}$$

The sum of them should be

$$\mathring{\boldsymbol{\sigma}} = \sum_i \mathring{\boldsymbol{\sigma}}(i) = \boldsymbol{\phi}\mathbf{T}_\beta \mathring{\boldsymbol{\beta}}. \tag{28}$$

In order to suppress the above ZEM(σ), a control strain $\boldsymbol{\varepsilon}_\Delta = \boldsymbol{\psi}_\Delta \boldsymbol{\alpha}_\Delta$ is employed and added to the primary one: $\boldsymbol{\varepsilon} = \boldsymbol{\psi}\boldsymbol{\alpha}$, and we have a modified strain trial function

$$\boldsymbol{\varepsilon}_m = \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_\Delta = [\boldsymbol{\psi} \;\; \boldsymbol{\psi}_\Delta] \begin{Bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}_\Delta \end{Bmatrix}. \tag{29}$$

Here the basis $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_\Delta$, followed by $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_\Delta$, are linearly independent of each other, hence the set of strains

$$\{\boldsymbol{\varepsilon}_m\} \equiv \mathrm{E}_m = \mathrm{E} \cup \mathrm{E}_\Delta, \quad \text{where } \mathrm{E}_\Delta \equiv \{\boldsymbol{\varepsilon}_\Delta\}.$$

Theorem 1. *Let $\mathring{\boldsymbol{\sigma}}$ be the ZEM(σ) appearing in the mixed element based on $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}_*)$. If $\boldsymbol{\varepsilon}_\Delta$ provides $\mathring{\boldsymbol{\sigma}}$ with an energy control:*

$$\forall \boldsymbol{\varepsilon}_\Delta \in \mathrm{E}_\Delta, \quad I_1(\boldsymbol{\varepsilon}_\Delta, \mathring{\boldsymbol{\sigma}}) = 0 \;\; \Rightarrow \;\; \mathring{\boldsymbol{\sigma}} = 0 \tag{30}$$

the modified mixed element based on $(\boldsymbol{\varepsilon}_m, \boldsymbol{\sigma}, \mathbf{u}_*)$ has no ZEM(σ).

Proof. Since $\boldsymbol{\varepsilon}_\Delta$ is independent of $\boldsymbol{\varepsilon}$, the energy constraint on $\boldsymbol{\sigma}$:

$$\forall (\boldsymbol{\varepsilon}_m, \mathbf{u}_*) \in \mathrm{E}_m \times U_*, \quad I_1(\boldsymbol{\varepsilon}_m, \boldsymbol{\sigma}) + I_2(\boldsymbol{\sigma}, \mathbf{u}_*) = 0 \tag{31}$$

is equivalent to

$$\begin{cases} \forall (\boldsymbol{\varepsilon}, \mathbf{u}_*) \in \mathrm{E} \times U_*, & I_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) + I_2(\boldsymbol{\sigma}, \mathbf{u}_*) = 0 \qquad (32a) \\ \forall \boldsymbol{\varepsilon}_\Delta \in \mathrm{E}_\Delta, & I_1(\boldsymbol{\varepsilon}_\Delta, \boldsymbol{\sigma}) = 0. \qquad (32b) \end{cases}$$

From (32a) we obtain the non-zero solution $\sigma = \mathring{\sigma}$ as exhibited by (28), and (32b) is then

$$\forall \varepsilon_\Delta \in E_\Delta, \quad I_1(\varepsilon_\Delta, \mathring{\sigma}) = 0.$$

With the introduction of the energy control (30) we have $\mathring{\sigma} = \sigma = 0$. The result is

$$\forall (\varepsilon_m, \mathbf{u}_*) \in E_m \times U_* \quad I_1(\varepsilon_m, \sigma) + I_2(\sigma, \mathbf{u}_*) = 0 \quad \Rightarrow \quad \sigma = 0, \tag{33}$$

i.e. the mixed element based on $(\varepsilon_m, \sigma, \mathbf{u}_*)$ has passed the stability condition (9) and has no ZEM(σ).

It is easy to choose a suitable control strain ε_Δ for $\mathring{\sigma}$ in (28). We denote

$$I_1(\varepsilon_\Delta, \mathring{\sigma}) = \alpha_\Delta^T \mathbf{F}_c \mathring{\beta}, \tag{34}$$

where the control matrix $\mathbf{F}_c$ is a square matrix of order n_0 if dim $(\alpha_\Delta) = n_0$. By regulating the strain basis function ψ_Δ such that $|\mathbf{F}_c| \neq 0$, then the energy control (30) can be achieved. A reliable trick for the choice of ψ_Δ may be suggested here. If $\mathring{\sigma} = \phi \mathbf{T}_\beta \mathring{\beta} = \mathring{\phi} \mathring{\beta}$, then it is only necessary to take $\varepsilon_\Delta = \mathring{\phi} \alpha_\Delta$. This is because the resulting control matrix

$$\mathbf{F}_c = \int_{v^e} \mathring{\phi}^T \mathring{\phi} \, dv,$$

must be a positive definite one.

5. ZEM(u) EXHIBITION AND CONTROL

When a 3-field mixed element does not pass the keeping rank condition (20), it has n_0 ZEM(u), and

$$n_0 = n_{q_*} - \text{rank}(\mathbf{G}_*). \tag{35}$$

We may denote the non-trivial solution of the homogeneous equation in (20) as

$$\mathbf{q}_* = \mathbf{T}_q \mathring{\mathbf{q}} = \mathbf{T}_q \sum_i^{n_0} \mathring{\mathbf{q}}(i). \tag{36}$$

In accordance with the displacement function (14) the ZEM(u) should be of the form

$$\mathring{\mathbf{u}}(i) = \mathbf{N}_* \mathbf{T}_q \mathring{\mathbf{q}}(i), \quad i = 1, 2, \ldots, n_0, \tag{37}$$

and the sum of them

$$\mathring{\mathbf{u}} = \sum_i \mathring{\mathbf{u}}(i) = \mathbf{N}_* \mathbf{T}_q \mathring{\mathbf{q}}. \tag{38}$$

In order to suppress the possible ZEM(u), a control stress $\sigma_\Delta \in \Sigma_\Delta \equiv \{\sigma_\Delta\}$ is introduced into the primary one: $\sigma \in \Sigma$. Of course σ_Δ and σ should be linearly independent of each other, and the modified stress

$$\sigma_m = \sigma + \sigma_\Delta = [\phi \ \ \phi_\Delta] \begin{Bmatrix} \beta \\ \beta_\Delta \end{Bmatrix} \in \Sigma \cup \Sigma_\Delta.$$

Theorem II. *Let* $\mathring{\mathbf{u}}$ *be the ZEM(u) which appear in the mixed element based on* $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}_*)$. *If* $\boldsymbol{\sigma}_\Delta$ *provided* $\mathring{\mathbf{u}}$ *an energy control:*

$$\forall \boldsymbol{\sigma}_\Delta \in \Sigma_\Delta, \quad I_2(\boldsymbol{\sigma}_\Delta, \mathring{\mathbf{u}}) = 0 \quad \Rightarrow \quad \mathring{\mathbf{u}} = 0 \tag{39}$$

the modified mixed element based on $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}_m, \mathbf{u}_*)$ *has no ZEM(u).*

Proof. The zero energy constrain on $\mathbf{u}_*$:

$$\forall \boldsymbol{\sigma}_m \in \Sigma \cup \Sigma_\Delta, \quad I_2(\boldsymbol{\sigma}_m, \mathbf{u}_*) = 0 \tag{40}$$

is equivalent to

$$\begin{cases} \forall \boldsymbol{\sigma} \in \Sigma, & I_2(\boldsymbol{\sigma}, \mathbf{u}_*) = 0 \quad (41a) \\ \forall \boldsymbol{\sigma}_\Delta \in \Sigma_\Delta, & I_2(\boldsymbol{\sigma}_\Delta, \mathbf{u}_*) = 0. \quad (41b) \end{cases}$$

From (41a) we obtain a non-zero solution $\mathbf{u}_* = \mathring{\mathbf{u}}$, and (41b) is then

$$\forall \boldsymbol{\sigma}_\Delta \in \Sigma_\Delta \quad I_2(\boldsymbol{\sigma}_\Delta, \mathring{\mathbf{u}}) = 0.$$

Under the energy control (39), $\mathring{\mathbf{u}} = \mathbf{u}_* = 0$. Finally,

$$\forall \boldsymbol{\sigma}_m \in \Sigma \cup \Sigma_\Delta, \quad I_2(\boldsymbol{\sigma}_m, \mathbf{u}_*) = 0 \quad \Rightarrow \quad \mathbf{u}_* = 0. \tag{42}$$

This means that the element based on $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}_m, \mathbf{u}_*)$ has passed the stability condition (10) and has no ZEM(u).

Let

$$\dim(\boldsymbol{\beta}_\Delta) = n_0 \quad \text{and} \quad I_2(\boldsymbol{\sigma}_\Delta, \mathring{\mathbf{u}}) = \boldsymbol{\beta}_\Delta^T \mathbf{G}_c \mathring{\mathbf{q}}. \tag{43}$$

Regulating the basis function $\boldsymbol{\phi}_\Delta$ of the control stress $\boldsymbol{\sigma}_\Delta$ such that $|\mathbf{G}_c| \neq 0$, then the energy control (39) can be achieved.

6. 3-FIELD HYBRID ELEMENT

Now we consider another kind of 3-field finite element—the 3-field hybrid element, for which, obviously, the definitions of ZEM in Section 2 are still valid. In the present case, like the strain parameter $\boldsymbol{\alpha}$, the stress parameter $\boldsymbol{\beta}$ will also be treated as a local parameter to be eliminated at the element level. Since $\boldsymbol{\beta}$ will no longer play the role of the basic discrete parameter, it is possible to set up an equilibrium equation for the hybrid element in terms of the displacement parameter $\mathbf{q}_*$ only.

In the mixed simultaneous eqn (17), the following relationship is included,

$$(\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})\boldsymbol{\beta} = \mathbf{G}_*\mathbf{q}_*. \tag{44}$$

The requirement for obtaining the unique solution of $\boldsymbol{\beta}$ is that the homogeneous equation concerning eqn (44) has only one trivial solution, i.e.

$$(\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})\boldsymbol{\beta} = 0 \quad \Rightarrow \quad \boldsymbol{\beta} = 0. \tag{45}$$

The matrix $\mathbf{A}^{-1}$ is a positive definite one, hence (45) is equivalent to the following condition:

$$\mathbf{F}\boldsymbol{\beta} = 0 \quad \Rightarrow \quad \boldsymbol{\beta} = 0. \tag{46}$$

In respect of the 3-field mixed element, (46) is an extra condition which ensures that the local parameter $\boldsymbol{\beta}$ can be completely determined by $\mathbf{q}_*$ and eliminated at the element level. It is found that (46) is sufficient to satisfy (19). Therefore, instead of the keeping rank condition of a mixed element (19), (46) should be a *keeping rank condition of a 3-field hybrid element*. Besides, it has been confirmed that another keeping rank condition of a mixed element (20) is still suitable for the present hybrid element.

Note that there exists a necessary condition for satisfying (46), i.e. $n_\alpha \geqslant n_\beta$. So the parameter matching condition (23) should be amended for a 3-field hybrid element, as

$$n_\alpha \geqslant n_\beta \geqslant n_{q_*}. \tag{47}$$

The condition (46) may equivalently be expressed by

$$\forall \boldsymbol{\alpha} \in \{\boldsymbol{\alpha}\}, \quad \boldsymbol{\alpha}^T\mathbf{F}\boldsymbol{\beta} = 0 \quad \Rightarrow \quad \boldsymbol{\beta} = 0,$$

or in an energy form

$$\forall \boldsymbol{\varepsilon} \in E, \quad I_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = 0 \quad \Rightarrow \quad \boldsymbol{\sigma} = 0. \tag{48}$$

Equation (48) together with (10) would be the *stability condition of the 3-field hybrid element*. Under condition (46) we have

$$\boldsymbol{\beta} = (\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})^{-1}\mathbf{G}_*\mathbf{q}_*. \tag{49}$$

By substitution of this into (18) we obtain the equilibrium equation of the hybrid element without rigid-body DOF as follows:

$$\mathbf{K}_*\mathbf{q}_* = [\mathbf{G}_*^T(\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})^{-1}\mathbf{G}_*]\mathbf{q}_* = 0. \tag{50}$$

The exhibition and control of the ZEM appearing in the hybrid elements on the whole are the same as those in the case of mixed elements except that when we determine the ZEM(σ) of a 3-field hybrid element by using formula (25), $\mathbf{T}_\beta\boldsymbol{\beta}^*$ is no longer the non-zero solution of the equation

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix}\boldsymbol{\beta} = 0 \quad \text{but of the equation} \quad \mathbf{F}\boldsymbol{\beta} = 0.$$

The key results obtained in the above sections are summed up in Table 1.

7. EXAMPLE

A simple but complete example is presented to illustrate the whole process of the ZEM analysis. We consider a $2a \times 2b$ rectangular 3-field hybrid element in Fig. 1 for the elastic plane problem, which possesses a constant strain, a constant stress and a bilinear displacement trial function:

Table 1. ZEM of the 3-field mixed and hybrid elements

Energy functional	$\pi = \frac{1}{2}\langle c\varepsilon, \varepsilon\rangle + l_1(\varepsilon, \sigma) + l_2(\sigma, \mathbf{u}_*) = \frac{1}{2}\alpha^T \mathbf{A}\alpha + \alpha^T \mathbf{F}\beta + \beta^T \mathbf{G}_* \mathbf{q}_*$	
Classification	3-field mixed element	3-field hybrid element
Element formulation (without rigid-body DOF)	$\begin{bmatrix} -\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F} & \mathbf{G}_* \\ \mathbf{G}_*^T & 0 \end{bmatrix} \begin{Bmatrix} \beta \\ \mathbf{q}_* \end{Bmatrix} = 0$ $\alpha = -\mathbf{A}^{-1}\mathbf{F}\beta$	$[G_*^T(\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})^{-1}\mathbf{G}_*]\mathbf{q}_* = 0$ $\beta = (\mathbf{F}^T\mathbf{A}^{-1}\mathbf{F})^{-1}\mathbf{G}_*\mathbf{q}_*$ $\alpha = -\mathbf{A}^{-1}\mathbf{F}\beta$
Element stability condition	$\begin{cases} \forall(\varepsilon, \mathbf{u}_*) \in \mathrm{E} \times U_*, \\ l_1(\varepsilon, \sigma) + l_2(\sigma, \mathbf{u}_*) = 0 \Rightarrow \sigma = 0 \\ \forall \sigma \in \Sigma, \quad l_2(\sigma, \mathbf{u}_*) = 0 \Rightarrow \mathbf{u}_* = 0 \end{cases}$	$\begin{cases} \forall \varepsilon \in \mathrm{E}, \quad l_1(\varepsilon, \sigma) = 0 \Rightarrow \sigma = 0 \\ \forall \sigma \in \Sigma, \quad l_2(\sigma, \mathbf{u}_*) = 0 \Rightarrow \mathbf{u}_* = 0 \end{cases}$
Keeping rank condition	$\begin{cases} \begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix} \beta = 0 \Rightarrow \beta = 0 \\ \mathbf{G}_*\mathbf{q}_* = 0 \Rightarrow \mathbf{q}_* = 0 \end{cases}$	$\begin{cases} \mathbf{F}\beta = 0 \Rightarrow \beta = 0 \\ \mathbf{G}_*\mathbf{q}_* = 0 \Rightarrow \mathbf{q}_* = 0 \end{cases}$
Parameter matching condition	$n_\alpha + n_{q_*} \geq n_\beta \geq n_{q_*}$	$n_\alpha \geq n_\beta \geq n_{q_*}$
ZEM(σ) formula and control	$\begin{cases} \dot{\sigma} = \phi \mathbf{T}_\beta \dot{\beta}, \text{ where} \\ \mathbf{T}_\beta \dot{\beta} = \text{non-zero solution of} \end{cases}$ $\begin{bmatrix} \mathbf{F} \\ \mathbf{G}_*^T \end{bmatrix} \beta = 0$ (Theorem I) Take a control strain ε_A, such that $l_2(\varepsilon_A, \dot{\sigma}) = 0 \Rightarrow \dot{\sigma} = 0$	$\dot{\sigma} = \phi \mathbf{T}_\beta \dot{\beta}$, where $\mathbf{T}_\beta \dot{\beta}$ = non-zero solution of $\mathbf{F}\beta = 0$ (Theorem I) Take a control strain ε_A, such that $l_1(\varepsilon_A, \dot{\sigma}) = 0 \Rightarrow \dot{\sigma} = 0$
ZEM(u) formula and control	$\begin{cases} \dot{\mathbf{u}} = N_* \mathbf{T}_q \dot{\mathbf{q}}, \text{ where} \\ \mathbf{T}_q \dot{\mathbf{q}} = \text{non-zero solution of} \end{cases}$ $\mathbf{G}_*\mathbf{q}_* = 0$ (Theorem II) Take a control stress σ_A, such that $l_2(\sigma_A, \dot{\mathbf{u}}) = 0 \Rightarrow \dot{\mathbf{u}} = 0$	$\begin{cases} \dot{\mathbf{u}} = \mathbf{N}_* \mathbf{T}_q \dot{\mathbf{q}}, \text{ where} \\ \mathbf{T}_q \dot{\mathbf{q}} = \text{non-zero solution of} \end{cases}$ $\mathbf{G}_*\mathbf{q}_* = 0$ (Theorem II) Take a control stress σ_A, such that $l_2(\sigma_A, \dot{\mathbf{u}}) = 0 \Rightarrow \dot{\mathbf{u}} = 0$

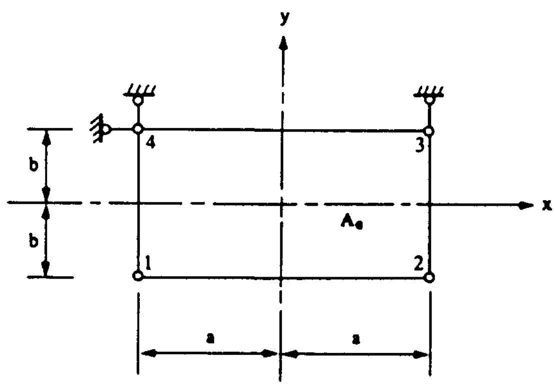

Fig. 1. Constrained rectangular 3-field hybrid element.

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \boldsymbol{\psi}\boldsymbol{\alpha} \tag{51}$$

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \boldsymbol{\phi}\boldsymbol{\beta} \tag{52}$$

and

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \mathbf{N}\mathbf{q}, \quad \mathbf{q} = \{u_1 \ v_1 \ \ldots \ u_4 \ v_4\}^{\mathrm{T}}. \tag{53}$$

The minimum nodal displacement constraints to prevent the element rigid-body motion are shown in Fig. 1. Accordingly, the element displacement (53) becomes

$$\mathbf{u}_* = \mathbf{N}_*\mathbf{q}_*, \quad \mathbf{q}_* = \{u_1 \ v_1 \ u_2 \ v_2 \ u_3\}^{\mathrm{T}}. \tag{54}$$

In accordance with the Hu–Washizu formulation

$$\pi = \int_{A^e} [\tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}} c \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}(\mathbf{D}\mathbf{u}_*)]\,\mathrm{d}A = \tfrac{1}{2}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{A}\boldsymbol{\alpha} + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{G}_*\mathbf{q}_*, \tag{55}$$

we have the element matrices

$$\mathbf{A} = \int_{A^e} \boldsymbol{\psi}^{\mathrm{T}} c \boldsymbol{\psi}\,\mathrm{d}A = 4abc, \tag{56}$$

$$\mathbf{F} = -\int_{A^e} \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\phi}\,\mathrm{d}A = -4ab \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \tag{57}$$

$$\mathbf{G}_* = \int_{A^e} \boldsymbol{\phi}^{\mathrm{T}}(\mathbf{D}\mathbf{N}_*)\,\mathrm{d}A = ab \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}. \tag{58}$$

Obviously, the keeping rank condition (46) is now satisfied, so that the hybrid element has no ZEM(σ). On the other hand, the homogeneous equation $\mathbf{G}_*\mathbf{q}_* = 0$ has the non-zero solution

$$\mathbf{q}_* = \mathbf{T}_q\mathring{\mathbf{q}} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ u_3 \end{Bmatrix}.$$

So the element has two ZEM(u), and they can be exhibited as follows:

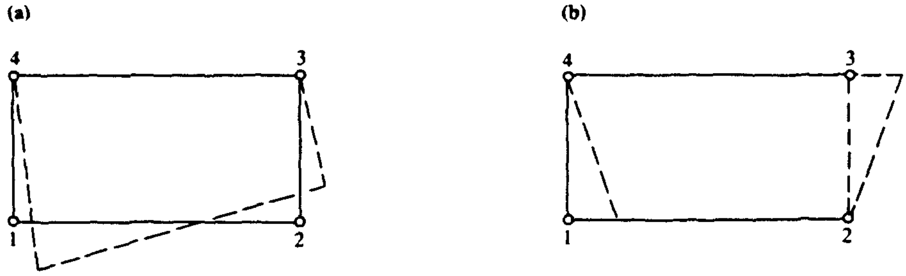

Fig. 2. ZEM(u) of the hybrid element based on ($\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}_*$).

$$\mathring{\mathbf{u}} = \begin{Bmatrix} \mathring{u} \\ \mathring{v} \end{Bmatrix} = \mathbf{N}_* \mathbf{T}_q \mathring{\mathbf{q}} = \frac{1}{2ab} \begin{bmatrix} b-y & ab+xy \\ x(b-y) & 0 \end{bmatrix} \mathring{\mathbf{q}}. \tag{59}$$

The related two ZEM(u) corresponding to

$$\mathring{\mathbf{q}}(1) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathring{\mathbf{q}}(2) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

are shown in Figs 2a and 2b respectively.

For the control of the above two ZEM(u) a control stress with two parameters is adopted here:

$$\boldsymbol{\varepsilon}_\Delta = \begin{Bmatrix} \sigma_x^\Delta \\ \sigma_y^\Delta \\ \sigma_{xy}^\Delta \end{Bmatrix} = \boldsymbol{\phi}_\Delta \boldsymbol{\beta}_\Delta, \quad \boldsymbol{\beta}_\Delta = \begin{Bmatrix} \beta_4 \\ \beta_5 \end{Bmatrix}. \tag{60}$$

There are many possible schemes for the choice of $\boldsymbol{\phi}_\Delta$, such as

$$\boldsymbol{\phi}_\Delta = \begin{bmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{bmatrix}, \tag{61}$$

or

$$\boldsymbol{\phi}_\Delta = \begin{bmatrix} y & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix}, \quad \text{etc.} \tag{62}$$

In terms of (61) and (59) the control matrix in (43) is

$$\mathbf{G}_c = \int_{A^e} \boldsymbol{\phi}_\Delta^{\mathrm{T}} (\mathbf{D}\mathbf{N}_* \mathbf{T}_q) \, \mathrm{d}A = 0,$$

and the scheme cannot be used to control ZEM(u). With the scheme (62),

$$\mathbf{G}_c = \begin{bmatrix} 0 & 2b^2/3 \\ -2a^2/3 & 0 \end{bmatrix}, \quad |\mathbf{G}_c| \neq 0.$$

So that the ZEM(u) exhibited in (59) will be controlled, and the modified stress trial function is of the form

$$\boldsymbol{\sigma}_{\mathrm{m}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}_\Delta = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_5 \end{Bmatrix} = \boldsymbol{\phi}_{\mathrm{m}} \boldsymbol{\beta}_{\mathrm{m}}. \tag{63}$$

It is notable that the requirement of $n_\alpha \geqslant n_\beta$ in the parameter matching condition (47) cannot be satisfied by the hybrid element based on $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{\mathrm{m}}, \mathbf{u}_*)$, and the element has at least $n_\beta - n_\alpha = 5-3 = 2$ ZEM(σ). Obviously, instead of (57), we now get

$$\mathbf{F} = -\int_{A^e} \boldsymbol{\phi}_{\mathrm{m}}^{\mathrm{T}} \boldsymbol{\psi}\, \mathrm{d}A = -4ab \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\mathbf{F}\boldsymbol{\beta}_{\mathrm{m}} = 0$ has the non-zero solution

$$\boldsymbol{\beta}_{\mathrm{m}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \beta_4 \\ \beta_5 \end{Bmatrix} = \mathbf{T}_\beta \mathring{\boldsymbol{\beta}},$$

and the ZEM(σ) would be

$$\mathring{\boldsymbol{\sigma}} = \begin{Bmatrix} \mathring{\sigma}_x \\ \mathring{\sigma}_y \\ \mathring{\sigma}_{xy} \end{Bmatrix} = \boldsymbol{\phi}_{\mathrm{m}} \mathbf{T}_\beta \mathring{\boldsymbol{\beta}} = \begin{bmatrix} y & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_4 \\ \beta_5 \end{Bmatrix} = \mathring{\boldsymbol{\phi}} \mathring{\boldsymbol{\beta}}. \tag{64}$$

Two ZEM(σ) corresponding to

$$\mathring{\boldsymbol{\beta}}(1) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathring{\boldsymbol{\beta}}(2) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

are shown in Fig. 3, and it is clear that they are orthogonal to the assumed constant strain

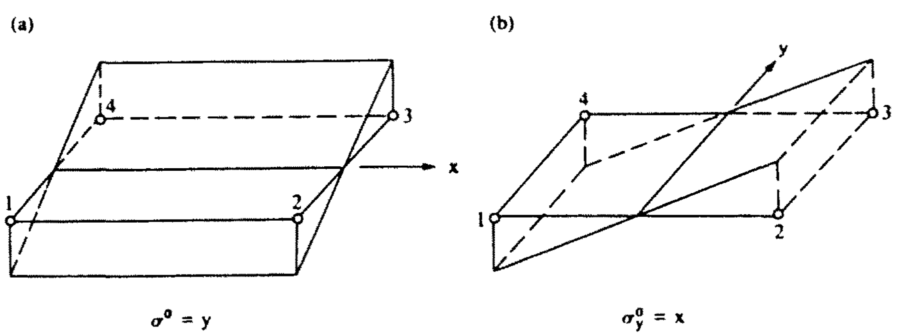

Fig. 3. ZEM(σ) of the hybrid element based on $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{\mathrm{m}}, \mathbf{u}_*)$.

(51), i.e. $\int_{A^e} \boldsymbol{\varepsilon}^T \mathring{\boldsymbol{\sigma}}\, dA = 0$. Thus the stress parameter $\boldsymbol{\beta}_m$ cannot be eliminated and should be solved with the displacement parameter $\mathbf{q}_*$ simultaneously. In other words, the element based on the trial function set $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}_m, \mathbf{u}_*)$ can only become a mixed one, not a hybrid one.

For the purpose of constructing a hybrid element the above two ZEM(σ) must be suppressed. According to Theorem I and in view of (64), we choose a control strain in such a manner,

$$\boldsymbol{\varepsilon}_\Delta = \begin{bmatrix} y & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \alpha_5 \end{Bmatrix} = \mathring{\boldsymbol{\phi}} \boldsymbol{\alpha}_\Delta. \tag{65}$$

Such that the control matrix in (34) takes the form

$$\mathbf{F}_c = -\int_{A^e} \mathring{\boldsymbol{\phi}}^T \mathring{\boldsymbol{\phi}}\, dA = -4ab \begin{bmatrix} b^2 & 0 \\ 0 & a^2 \end{bmatrix}. \tag{66}$$

$|\mathbf{F}_c| \neq 0$, and the ZEM(σ) have been controlled. Denoting the modified strain as:

$$\boldsymbol{\varepsilon}_m = \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_\Delta = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_5 \end{Bmatrix} = \boldsymbol{\phi}_m \boldsymbol{\alpha}_m, \tag{67}$$

then we can state that the 3-field hybrid element based on $(\boldsymbol{\varepsilon}_m, \boldsymbol{\sigma}_m, \mathbf{u}_*)$ has neither ZEM(u) nor ZEM(σ) and must be a stable element. Actually, it has been confirmed that this 3-field hybrid element is equivalent to Pian's 5β stress element (Pian, 1964) which is based on $(\boldsymbol{\sigma}_m, \mathbf{u}_*)$ and the modified complementary energy/Reissner principle.

8. DISCUSSIONS AND CONCLUSIONS

A theory for ZEM for 3-field finite elements has been proposed in this paper. A systematic study was made on mixed elements and hybrid elements based on Hu–Washizu principle. Further discussions and explanations will be made here with regard to the following three problems:

(a) With regard to the convergence problem of mixed/hybrid elements, mechanicians and mathematicians tend to treat the matter differently, apart from the fundamental requirement of the completeness of the trial functions. The latter used a rigorous mathematical expression in the form of an inf–sup condition with the constant independent of h to be a necessary and sufficient condition for convergence and uniqueness, while the former approached the problem from energy principles, and regarded that the imposition of certain requirements would guarantee the convergence of the solution of a discretized problem. They are
   (i) stability requirements—Hu (1990), stated that the trial functions of the field variables (generalized forces and generalized displacements) must be capable of working. In other words, the trial functions should not include any ZEM which does not contribute towards the energy functional of the system. Furthermore, the weak continuity conditions between elements should be satisfied in order to guarantee that relative rigid body movement exists between elements (Wu and Cheung, 1991), and
   (ii) compatibility requirements—for an element with non-conforming trial functions for both the stresses and displacements, the energy compatibility condition proposed by Wu and Bufler (1991) must be satisfied. This stipulates that the work done by the non-conforming component $\mathbf{u}_\lambda$ of the element displacement trial func-

tions and the element surface force $\mathbf{T} = \mathbf{T}(\sigma)$ should be zero, i.e.

$$W = \sum_e \oint_{\partial\Gamma^e} \mathbf{T}^T \mathbf{u}_\lambda \, ds = 0. \tag{68}$$

Otherwise W should be deleted from the energy functional.

In this discussion, we have only considered the ZEM problem connected with the stability requirement, not the compatibility requirement.

(b) The proposed ZEM analysis is valid for a single typical element. However, it must be pointed out that satisfying the element stability condition/element keeping rank condition can only guarantee that the element will have no ZEM, it will in no way guarantee that a system of elements will have no ZEM. For a mixed element system there is no straightforward solution since the displacement continuity condition between elements can suppress ZEM(u) but at the same time introduce new ZEM (σ), while for the stress continuity condition, the reverse is true. The best solution appears to be the carrying-out of a patch stability test, since it would be reasonable to say that if the patch of elements has no ZEM, then the whole discretized system would also have no ZEM. This is in line with the so-called local inf–sup condition with the constant independent of h in mathematics and the mathematical basis for such a patch test was proposed by Zhou (1986). The implementation method and the formulae for the patch stability test of 2-field elements has already been given by Wu (1990), and there should be no difficulty in extending the formulae to a 3-field element situation. For example, if a patch of element is regarded as a single element, then the keeping rank conditions (19) and (20) for the 3-field mixed elements will become the formulae for the stability test of the patch of elements.

(c) It is common practice to introduce internal parameters λ (to define the element's internal displacements) in order to improve the numerical characteristics of an element (Cheung and Chen, 1988; Chen and Cheung, 1987). The internal displacement field is then

$$\mathbf{u}_\lambda = \mathbf{N}_\lambda \boldsymbol{\lambda}$$

and the element energy function (15) must be revised to

$$\pi = \tfrac{1}{2}\boldsymbol{\alpha}^T \mathbf{A} \boldsymbol{\alpha} + \boldsymbol{\alpha}^T \mathbf{F} \boldsymbol{\beta} + \boldsymbol{\beta}^T [\mathbf{G}_* \;\; \mathbf{G}_\lambda] \begin{Bmatrix} \mathbf{q}_* \\ \boldsymbol{\lambda} \end{Bmatrix}. \tag{69}$$

Fortunately, no change is necessary in the ZEM analysis for this type of element except that the following modifications will have to be observed:

$$\mathbf{q}_* \to \begin{Bmatrix} \mathbf{q}_* \\ \boldsymbol{\lambda} \end{Bmatrix},$$

$$\mathbf{G}_* \to [\mathbf{G}_* \;\; \mathbf{G}_\lambda]$$

and

$$n_{q_*} \to n_{q_*} + n_\lambda \quad (n_\lambda = \dim(\boldsymbol{\lambda})).$$

However, if we wish to eliminate λ at the element level, then the following additional condition must also be satisfied

$$\mathbf{G}_\lambda \boldsymbol{\lambda} = 0 \quad \Rightarrow \quad \boldsymbol{\lambda} = 0. \tag{70}$$

It is not difficult to prove that (70) is in fact a sufficient condition to guarantee that static condensation can be effected for the mixed/hybrid elements with internal displacements.

(d) The ZEM analysis presented in this paper has been made for a single element. For a discrete system it would be necessary to consider the effect of element assembly and then carry out a patch stability test (Wu, 1990) in accordance with Table 1. The above observation is certainly true for mixed elements in which both the stress and displacement are continuous at the element interface or the common nodes of elements. However, in the case of 3-field hybrid elements, because the continuity of displacements at the element interface or the common nodes of elements can only prevent the appearance of additional ZEM(u), it is possible to conclude that provided the parameter matching condition (47) has been satisfied for the various combination of elements, the stability for a discrete system can in fact be guaranteed by observing the keeping rank conditions ((20) and (46)) of an individual hybrid element.

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